POSITIVE DEHN TWIST EXPRESSION FOR A $\mathbb{Z}_3$ ACTION ON Σ_g

HISAAKI ENDO AND YUSUF Z. GURTAS

ABSTRACT. A positive Dehn twist product for a $\mathbb{Z}_3$ action on the 2-dimensional closed, compact, oriented surface Σ_g is presented. The homeomorphism invariants of the resulting symplectic 4-manifolds are computed.

INTRODUCTION

This article attempts to answer a question raised by Feng Luo in [6] which asks for a Dehn twist expression for the generator of a $\mathbb{Z}_3$ action with $g+2$ fixed points on the 2-dimensional closed, compact, oriented surface Σ_g. By the work of Nielsen, there is only one such action on Σ_g, [6].

In Section 2 we build a closed genus g surface Σ_g using g copies of tori with boundary as building blocks in order to realize that action on Σ_g. We simply take an order three element from the mapping class group $\mathcal{M}_1$ of torus and juxtapose its Dehn twist expression in $\mathcal{M}_g$, considering torus with boundary as a subsurface of Σ_g and taking the orientation into consideration in the gluing process. We start with a torus with one boundary component oriented positively. Then glue a torus with two boundary components oriented negatively to it. Then keep adding more tori with boundary with alternating orientations and finally cap it off with a torus with one boundary component oppositely oriented as the previous copy. We aim at a Dehn twist product for the generator of the $\mathbb{Z}_3$ action on Σ_g that uses only positive exponents in order to make sure that the 4-manifold it defines as Lefschetz fibration carries symplectic structure. This becomes a challenge because the negatively oriented bounded tori introduce into the expression many elements with negative exponents and there are still some negative powers to be eliminated from the expression for genus $g > 6$. Therefore this work is still in progress.

In Section 3 we show explicitly how to obtain a positive Dehn twist product for the generator of the $\mathbb{Z}_3$ action on $\Sigma_g, g \leq 6$. What seems to be working for low genus doesn't generalize to higher genus easily and the construction evolves rather ad hoc, at least partially.

In Section 4 we compute the Euler characteristic and signatures of the 4-manifolds given by the words that are obtained in Section 3. The method introduced by the first author and S.Nagami is used for signature computations, [1].

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1. Review of Relations in $\mathcal{M}_1, \mathcal{M}_1^1$ and $\mathcal{M}_1^2$

The mapping class group $\mathcal{M}_1$ of torus is generated by Dehn twists about the cycles α and β, Figure 1, subject to the relations

$$\begin{aligned} \alpha\beta\alpha &= \beta\alpha\beta \\ (\alpha\beta)^6 &= 1. \end{aligned} \tag{1.1}$$

Here, by abuse of notation, we use α and β to mean Dehn twists about them for simplicity. The first relation is called *braid relation* and it exists between every pair of curves that intersect transversely.

Torus with one, two, and three boundary components are subject to the relations

$$(\alpha\beta)^6 = \delta \quad \text{and} \quad (\beta\alpha\beta\gamma)^3 = (\alpha\beta\gamma)^4 = \delta_1\delta_2, \quad \text{and} \quad (\alpha_1\alpha_2\alpha_3\beta)^3 = \delta_1\delta_2\delta_3 \tag{1.2}$$

respectively. The last one is also called *star relation*, [3].

The basic idea that is used in this paper is to glue several copies of torus with

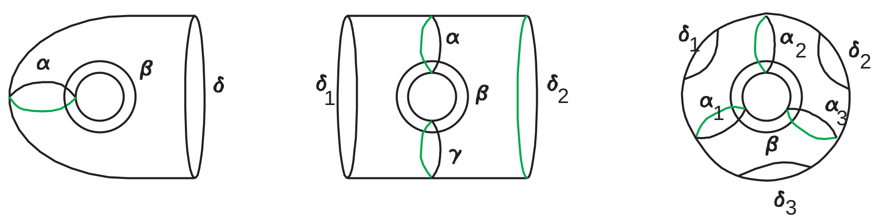


Figure 1.

two boundary components together and cap the resulting bounded surface off with two copies of torus with one boundary component, one on each end, to get a closed surface of genus g. We take the word

$$(\alpha\beta)^2 \tag{1.3}$$

on the two end copies and the word

$$\beta\alpha\beta\gamma \tag{1.4}$$

on each of the remaining copies in between and juxtapose them with alternating signs to come up with an order three element in the mapping class group of the resulting closed genus g surface.

2. Construction of the order three element on Σ_g

In this section we will construct an order three element in the mapping class group of closed genus g surface using the words (1.3) and (1.4) according to their position in the gluing process. First case is when genus g is even.

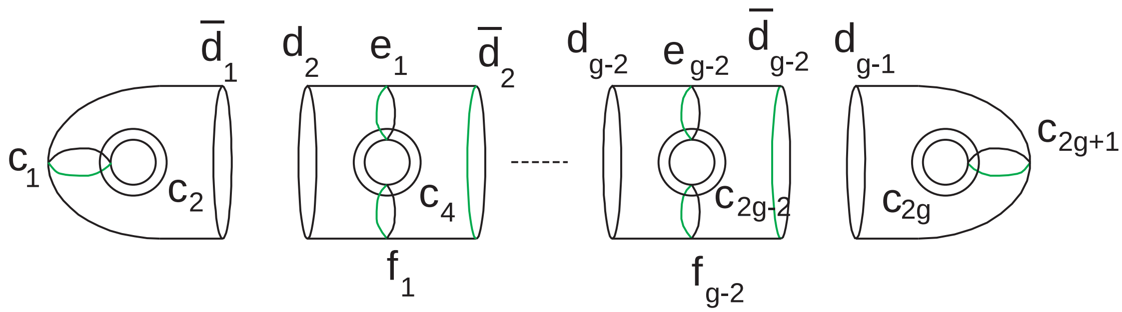


FIGURE 2.

2.1. **Genus g-even.** We juxtapose the words of type (1.3) and (1.4) on each of the bounded surfaces above by paying careful attention to the orientation:

$$
\begin{gathered}
(c_1c_2)^2 \\
(c_4e_1c_4f_1)^{-1} \\
c_6e_2c_6f_2 \\
(c_8e_3c_8f_3)^{-1} \\
\vdots \\
(c_{2g-4}e_{g-3}c_{2g-4}f_{g-3})^{-1} \\
c_{2g-2}e_{g-2}c_{2g-2}f_{g-2} \\
(c_{2g+1}c_{2g})^{-2}
\end{gathered}
\tag{2.1}
$$

Every other surface will be negatively oriented so that we can glue the boundaries together. Using the chain relation

$$(c_{2i+2}e_ic_{2i+2}f_i)^3 = d_i\bar{d}_i$$

on torus with two boundary components and

$$(c_{2g+1}c_{2g})^6 = d_{g-1}$$

on torus with one boundary component, the expressions containing negative exponents in (2.1) can be written as

$$
\begin{aligned}
(c_4e_1c_4f_1)^{-1} &= d_2^{-1}\,(c_4e_1c_4f_1)^2\,\bar{d}_2^{-1} \\
(c_8e_3c_8f_3)^{-1} &= d_3^{-1}\,(c_8e_3c_8f_3)^2\,\bar{d}_3^{-1} \\
&\vdots \\
(c_{2g-4}e_{g-3}c_{2g-4}f_{g-3})^{-1} &= d_{g-3}^{-1}\,(c_{2g-4}e_{g-3}c_{2g-4}f_{g-3})^2\,\bar{d}_{g-3}^{-1}
\end{aligned}
\tag{2.2}
$$

and the last one as

$$(c_{2g+1}c_{2g})^{-2} = d_{g-1}^{-1}\,(c_{2g+1}c_{2g})^4\,. \tag{2.3}$$

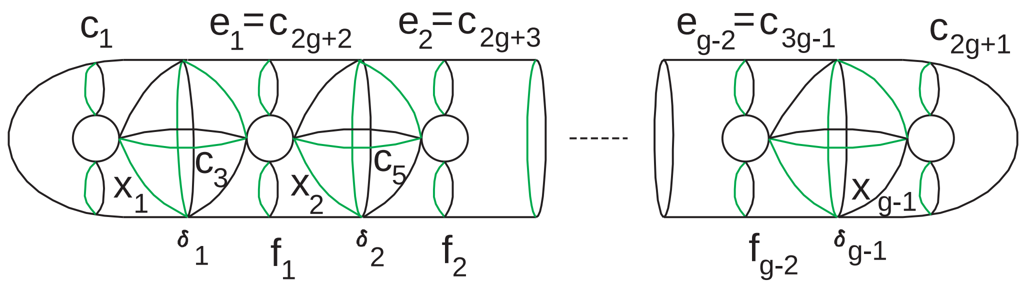


FIGURE 3.

We glue the bounded surfaces together and use the lantern relations

$$
\begin{aligned}
\delta_1 x_1 c_3 &= c_1 c_1 e_1 f_1 \\
\delta_2 x_2 c_5 &= e_1 f_1 e_2 f_2 \\
&\vdots \\
\delta_{g-2} x_{g-2} c_{2g-3} &= e_{g-3} f_{g-3} e_{g-2} f_{g-2} \\
\delta_{g-1} x_{g-1} c_{2g-1} &= e_{g-2} f_{g-2} c_{2g+1} c_{2g+1}
\end{aligned} \tag{2.4}
$$

to eliminate the negative exponents of $\overline{d}_i$ and d_i in (2.2) and (2.3) using the fact that $\overline{d}_i = d_{i+1} = \delta_i$ after gluing. Solving (2.4) for δ_i^{-1} we get

$$
\begin{aligned}
\delta_1^{-1} &= x_1 c_3 c_1^{-1} c_1^{-1} e_1^{-1} f_1^{-1} \\
\delta_2^{-1} &= x_2 c_5 e_1^{-1} f_1^{-1} e_2^{-1} f_2^{-1} \\
&\vdots \\
\delta_{g-2}^{-1} &= x_{g-2} c_{2g-3} e_{g-3}^{-1} f_{g-3}^{-1} e_{g-2}^{-1} f_{g-2}^{-1} \\
\delta_{g-1}^{-1} &= x_{g-1} c_{2g-1} e_{g-2}^{-1} f_{g-2}^{-1} c_{2g+1}^{-1} c_{2g+1}^{-1}
\end{aligned} \tag{2.5}
$$

Therefore equations (2.2) and (2.3) become

$$
\begin{aligned}
(c_4 e_1 c_4 f_1)^{-1} &= x_1 c_3 c_1^{-1} c_1^{-1} e_1^{-1} f_1^{-1} \, (c_4 e_1 c_4 f_1)^2 \, x_2 c_5 e_1^{-1} f_1^{-1} e_2^{-1} f_2^{-1} \\
(c_8 e_3 c_8 f_3)^{-1} &= x_3 c_7 e_2^{-1} f_2^{-1} e_3^{-1} f_3^{-1} \, (c_8 e_3 c_8 f_3)^2 \, x_4 c_9 e_3^{-1} f_3^{-1} e_4^{-1} f_4^{-1} \\
&\vdots \\
(c_{2g-4} e_{g-3} c_{2g-4} f_{g-3})^{-1} &= x_{g-3} c_{2g-5} e_{g-4}^{-1} f_{g-4}^{-1} e_{g-3}^{-1} f_{g-3}^{-1} \, (c_{2g-4} e_{g-3} c_{2g-4} f_{g-3})^2 \, x_{g-2} c_{2g-3} e_{g-3}^{-1} f_{g-3}^{-1} e_{g-2}^{-1} f_{g-2}^{-1}
\end{aligned} \tag{2.6}
$$

and

$$
(c_{2g+1} c_{2g})^{-2} = x_{g-1} c_{2g-1} e_{g-2}^{-1} f_{g-2}^{-1} c_{2g+1}^{-1} c_{2g+1}^{-1} \, (c_{2g+1} c_{2g})^4 , \tag{2.7}
$$

and (2.1) becomes

$$
\begin{gathered}
(c_1 c_2)^2 \\
x_1 c_3 c_1^{-1} c_1^{-1} e_1^{-1} f_1^{-1} \, (c_4 e_1 c_4 f_1)^2 \, x_2 c_5 e_1^{-1} f_1^{-1} e_2^{-1} f_2^{-1} \\
c_6 e_2 c_6 f_2
\end{gathered}
$$

$$
\begin{gathered}
x_3 c_7 e_2^{-1} f_2^{-1} e_3^{-1} f_3^{-1} \left(c_8 e_3 c_8 f_3\right)^2 x_4 c_9 e_3^{-1} f_3^{-1} e_4^{-1} f_4^{-1} \\
c_{10} e_4 c_{10} f_4 \\
\vdots \\
c_{2g-6} e_{g-4} c_{2g-6} f_{g-4} \\
x_{g-3} c_{2g-5} e_{g-4}^{-1} f_{g-4}^{-1} e_{g-3}^{-1} f_{g-3}^{-1} \left(c_{2g-4} e_{g-3} c_{2g-4} f_{g-3}\right)^2 x_{g-2} c_{2g-3} e_{g-3}^{-1} f_{g-3}^{-1} e_{g-2}^{-1} f_{g-2}^{-1} \\
c_{2g-2} e_{g-2} c_{2g-2} f_{g-2} \\
x_{g-1} c_{2g-1} e_{g-2}^{-1} f_{g-2}^{-1} c_{2g+1}^{-1} c_{2g+1}^{-1} \left(c_{2g+1} c_{2g}\right)^4 .
\end{gathered} \tag{2.8}
$$

Juxtaposing these words, we obtain

$$
\begin{gathered}
\left(c_1 c_2\right)^2 x_1 c_3 c_1^{-1} c_1^{-1} e_1^{-1} f_1^{-1} \left(c_4 e_1 c_4 f_1\right)^2 x_2 c_5 e_1^{-1} f_1^{-1} e_2^{-1} f_2^{-1} c_6 e_2 c_6 f_2 \\
x_3 c_7 e_2^{-1} f_2^{-1} e_3^{-1} f_3^{-1} \left(c_8 e_3 c_8 f_3\right)^2 x_4 c_9 e_3^{-1} f_3^{-1} e_4^{-1} f_4^{-1} c_{10} e_4 c_{10} f_4 \\
x_5 c_{11} e_4^{-1} f_4^{-1} e_5^{-1} f_5^{-1} \left(c_{12} e_5 c_{12} f_5\right)^2 x_6 c_{13} e_5^{-1} f_5^{-1} e_6^{-1} f_6^{-1} c_{14} e_6 c_{14} f_6 \\
\vdots \\
c_{2g-6} e_{g-4} c_{2g-6} f_{g-4} x_{g-3} c_{2g-5} e_{g-4}^{-1} f_{g-4}^{-1} e_{g-3}^{-1} f_{g-3}^{-1} c_{2g-4} e_{g-3} c_{2g-4} f_{g-3} c_{2g-4} e_{g-3} c_{2g-4} f_{g-3} \\
x_{g-2} c_{2g-3} e_{g-3}^{-1} f_{g-3}^{-1} e_{g-2}^{-1} f_{g-2}^{-1} c_{2g-2} e_{g-2} c_{2g-2} f_{g-2} x_{g-1} c_{2g-1} e_{g-2}^{-1} f_{g-2}^{-1} c_{2g+1}^{-1} c_{2g+1}^{-1} \left(c_{2g+1} c_{2g}\right)^4
\end{gathered} \tag{2.9}
$$

Next, we will eliminate the negative exponents using braid and commutativity relations only. Let's expand the parenthesis in the top three lines in (2.9):

$$
\begin{gathered}
c_1 \underline{c_2 c_1 c_2} x_1 c_3 c_1^{-1} c_1^{-1} e_1^{-1} f_1^{-1} \underline{c_4 e_1 c_4} f_1 \underline{c_4 e_1 c_4} f_1 x_2 c_5 e_1^{-1} f_1^{-1} e_2^{-1} f_2^{-1} \underline{c_6 e_2 c_6} f_2 \\
x_3 c_7 e_2^{-1} f_2^{-1} e_3^{-1} f_3^{-1} \underline{c_8 e_3 c_8} f_3 \underline{c_8 e_3 c_8} f_3 x_4 c_9 e_3^{-1} f_3^{-1} e_4^{-1} f_4^{-1} \underline{c_{10} e_4 c_{10}} f_4 \\
x_5 c_{11} e_4^{-1} f_4^{-1} e_5^{-1} f_5^{-1} \underline{c_{12} e_5 c_{12}} f_5 \underline{c_{12} e_5 c_{12}} f_5 x_6 c_{13} e_5^{-1} f_5^{-1} e_6^{-1} f_6^{-1} \underline{c_{14} e_6 c_{14}} f_6
\end{gathered}
$$

and use braid relation for the underlined triples:

$$
\begin{gathered}
c_1 c_1 c_2 \underline{c_1} x_1 c_3 \underline{c_1}^{-1} c_1^{-1} \underline{e_1}^{-1} f_1^{-1} \underline{e_1} c_4 e_1 f_1 e_1 c_4 \underline{e_1 f_1} x_2 c_5 \underline{e_1^{-1} f_1}^{-1} \underline{e_2}^{-1} f_2^{-1} \underline{e_2} c_6 \underline{e_2 f_2} \\
x_3 c_7 \underline{e_2^{-1} f_2}^{-1} \underline{e_3}^{-1} f_3^{-1} \underline{e_3} c_8 e_3 f_3 e_3 c_8 \underline{e_3 f_3} x_4 c_9 \underline{e_3^{-1} f_3}^{-1} \underline{e_4}^{-1} f_4^{-1} \underline{e_4} c_{10} \underline{e_4 f_4} \\
x_5 c_{11} \underline{e_4^{-1} f_4}^{-1} \underline{e_5}^{-1} f_5^{-1} \underline{e_5} c_{12} e_5 f_5 e_5 c_{12} \underline{e_5 f_5} x_6 c_{13} \underline{e_5^{-1} f_5}^{-1} \underline{e_6}^{-1} f_6^{-1} \underline{e_6} c_{14} e_6 f_6 .
\end{gathered} \tag{2.10}
$$

Next, cancel the underlined pairs above using commutativity:

$$
c_1 c_1 c_2 x_1 c_3 c_1^{-1} f_1^{-1} c_4 e_1 f_1 e_1 c_4 x_2 c_5 f_2^{-1} c_6 x_3 c_7 f_3^{-1} c_8 e_3 f_3 e_3 c_8 x_4 c_9 f_4^{-1} c_{10} x_5 c_{11} f_5^{-1} c_{12} e_5 f_5 e_5 c_{12} x_6 c_{13} f_6^{-1} c_{14} e_6 f_6
$$

Now, using commutativity and the fact that $t_{f(\alpha)} = f t_\alpha f^{-1}$, for any simple closed curve α in Σ_g and any diffeomorphism $f : \Sigma_g \to \Sigma_g$, where t_α and $t_{f(\alpha)}$ are Dehn twists about the curves α and $f(\alpha)$ respectively, we can write

$$c_1c_1c_2c_1^{-1}x_1c_3f_1^{-1}c_4f_1e_1e_1c_4x_2c_5f_2^{-1}c_6x_3c_7f_3^{-1}c_8f_3e_3e_3c_8x_4c_9f_4^{-1}c_{10}x_5c_{11}f_5^{-1}c_{12}f_5e_5e_5c_{12}x_6c_{13}f_6^{-1}c_{14}$$

as

(2.11)

$$c_1dx_1c_3r_1e_1e_1c_4x_2c_5f_2^{-1}c_6x_3c_7r_3e_3e_3c_8x_4c_9f_4^{-1}c_{10}x_5c_{11}r_5e_5e_5c_{12}x_6c_{13}f_6^{-1}c_{14}x_7c_{15},$$

where $d = c_1c_2c_1^{-1}$ and $r_i = f_i^{-1}c_{2i+2}f_i, i = 1, 3, 5$.

The last portion of the word in (2.9) will be simplified using the same procedure:

$$\begin{gathered}
\underline{c_{2g-6}e_{g-4}c_{2g-6}}f_{g-4}x_{g-3}c_{2g-5}e_{g-4}^{-1}f_{g-4}^{-1}e_{g-3}^{-1}f_{g-3}^{-1}\underline{c_{2g-4}e_{g-3}c_{2g-4}}f_{g-3}\underline{c_{2g-4}e_{g-3}c_{2g-4}}f_{g-3}\\
x_{g-2}c_{2g-3}e_{g-3}^{-1}f_{g-3}^{-1}e_{g-2}^{-1}f_{g-2}^{-1}\underline{c_{2g-2}e_{g-2}c_{2g-2}}f_{g-2}x_{g-1}c_{2g-1}e_{g-2}^{-1}f_{g-2}^{-1}c_{2g+1}^{-1}\underline{c_{2g+1}^{-1}c_{2g+1}}c_{2g}\left(c_{2g+1}c_{2g}\right)^3\\
\downarrow\\
e_{g-4}c_{2g-6}\underline{e_{g-4}f_{g-4}}x_{g-3}c_{2g-5}\underline{e_{g-4}^{-1}f_{g-4}}^{-1}\underline{e_{g-3}}^{-1}f_{g-3}^{-1}\underline{e_{g-3}}c_{2g-4}e_{g-3}f_{g-3}e_{g-3}c_{2g-4}\underline{e_{g-3}f_{g-3}}\\
x_{g-2}c_{2g-3}\underline{e_{g-3}^{-1}f_{g-3}}^{-1}\underline{e_{g-2}}^{-1}f_{g-2}^{-1}\underline{e_{g-2}}c_{2g-2}\underline{e_{g-2}f_{g-2}}x_{g-1}c_{2g-1}\underline{e_{g-2}^{-1}f_{g-2}}^{-1}c_{2g+1}^{-1}\underline{c_{2g}c_{2g+1}c_{2g}}\left(c_{2g+1}c_{2g}\right)^2\\
\downarrow\\
c_{2g-6}x_{g-3}c_{2g-5}f_{g-3}^{-1}c_{2g-4}\underline{e_{g-3}f_{g-3}}e_{g-3}c_{2g-4}x_{g-2}c_{2g-3}f_{g-2}^{-1}\\
c_{2g-2}x_{g-1}c_{2g-1}\underline{c_{2g+1}^{-1}c_{2g+1}}c_{2g}c_{2g+1}\left(c_{2g+1}c_{2g}\right)^2\\
\downarrow\\
c_{2g-6}x_{g-3}c_{2g-5}f_{g-3}^{-1}c_{2g-4}f_{g-3}e_{g-3}e_{g-3}c_{2g-4}x_{g-2}c_{2g-3}f_{g-2}^{-1}\\
c_{2g-2}x_{g-1}c_{2g-1}c_{2g}c_{2g+1}\left(c_{2g+1}c_{2g}\right)^2\\
\downarrow
\end{gathered}$$

(2.12)

$$c_{2g-6}x_{g-3}c_{2g-5}r_{g-3}e_{g-3}e_{g-3}c_{2g-4}x_{g-2}c_{2g-3}f_{g-2}^{-1}c_{2g-2}x_{g-1}c_{2g-1}c_{2g}c_{2g+1}c_{2g+1}c_{2g}c_{2g+1}c_{2g},$$

where $r_{g-3} = f_{g-3}^{-1}c_{2g-4}f_{g-3}$.

Combining (2.11) and (2.12) we obtain

$$\begin{gathered}
c_1dx_1c_3r_1e_1e_1c_4x_2c_5f_2^{-1}c_6x_3c_7r_3e_3e_3c_8x_4c_9f_4^{-1}c_{10}x_5c_{11}r_5e_5e_5c_{12}x_6c_{13}f_6^{-1}c_{14}x_7c_{15}\\
\vdots\\
c_{2g-6}x_{g-3}c_{2g-5}r_{g-3}e_{g-3}e_{g-3}c_{2g-4}x_{g-2}c_{2g-3}f_{g-2}^{-1}c_{2g-2}x_{g-1}c_{2g-1}c_{2g}c_{2g+1}c_{2g+1}c_{2g}c_{2g+1}c_{2g}
\end{gathered} \tag{2.13}$$

It seems to be a little challenging to remove the remaining negative exponents from this last expression at this point.

In a more compact form the word can be written as:

$$c_1dx_1c_3r_1e_1e_1c_4x_2c_5f_2^{-1}W_6W_8\cdots W_gc_{2g-2}x_{g-1}c_{2g-1}c_{2g}c_{2g+1}c_{2g+1}c_{2g}c_{2g+1}c_{2g},$$

where $W_i = c_{2i-6}x_{i-3}c_{2i-5}r_{i-3}e_{i-3}e_{i-3}c_{2i-4}x_{i-2}c_{2i-3}f_{i-2}^{-1}, i = 6, 8, \ldots, g$.

2.2. **Genus g-odd.** Most of the argument will be similar to the even case; we just need to make some changes on the indices.

The following are the words from each component listed with alternating signs

$$\begin{gathered}(c_1c_2)^2\\ (c_4e_1c_4f_1)^{-1}\\ c_6e_2c_6f_2\\ \vdots\\ (c_{2g-2}e_{g-2}c_{2g-2}f_{g-2})^{-1}\\ (c_{2g+1}c_{2g})^2.\end{gathered} \tag{2.14}$$

Now, (2.2) becomes

$$\begin{aligned}(c_4e_1c_4f_1)^{-1} &= d_2^{-1}\,(c_4e_1c_4f_1)^2\,\overline{d}_2^{-1}\\ (c_8e_3c_8f_3)^{-1} &= d_3^{-1}\,(c_8e_3c_8f_3)^2\,\overline{d}_3^{-1}\\ &\vdots\\ (c_{2g-2}e_{g-2}c_{2g-2}f_{g-2})^{-1} &= d_{g-2}^{-1}\,(c_{2g-2}e_{g-2}c_{2g-2}f_{g-2})^2\,\overline{d}_{g-2}^{-1}.\end{aligned} \tag{2.15}$$

(2.4) and (2.5) are still the same. Therefore (2.6) becomes

$$\begin{aligned}(c_4e_1c_4f_1)^{-1} &= x_1c_3c_1^{-1}c_1^{-1}e_1^{-1}f_1^{-1}\,(c_4e_1c_4f_1)^2\,x_2c_5e_1^{-1}f_1^{-1}e_2^{-1}f_2^{-1}\\ (c_8e_3c_8f_3)^{-1} &= x_3c_7e_2^{-1}f_2^{-1}e_3^{-1}f_3^{-1}\,(c_8e_3c_8f_3)^2\,x_4c_9e_3^{-1}f_3^{-1}e_4^{-1}f_4^{-1}\\ &\vdots\\ (c_{2g-2}e_{g-2}c_{2g-2}f_{g-2})^{-1} &= x_{g-2}c_{2g-3}e_{g-3}^{-1}f_{g-3}^{-1}e_{g-2}^{-1}f_{g-2}^{-1}\,(c_{2g-2}e_{g-2}c_{2g-2}f_{g-2})^2x_{g-1}c_{2g-1}e_{g-2}^{-1}f_{g-2}^{-1}c_{2g+1}^{-1}c_{2g+1}^{-1}\end{aligned} \tag{2.16}$$

using lantern relations in (2.15) to replace d_i^{-1} and $\overline{d}_i^{-1}$.

Then (2.8) becomes

$$\begin{gathered}(c_1c_2)^2\\ x_1c_3c_1^{-1}c_1^{-1}e_1^{-1}f_1^{-1}\,(c_4e_1c_4f_1)^2\,x_2c_5e_1^{-1}f_1^{-1}e_2^{-1}f_2^{-1}\\ c_6e_2c_6f_2\\ x_3c_7e_2^{-1}f_2^{-1}e_3^{-1}f_3^{-1}\,(c_8e_3c_8f_3)^2\,x_4c_9e_3^{-1}f_3^{-1}e_4^{-1}f_4^{-1}\\ c_{10}e_4c_{10}f_4\\ \vdots\\ c_{2g-4}e_{g-3}c_{2g-4}f_{g-3}\\ x_{g-2}c_{2g-3}e_{g-3}^{-1}f_{g-3}^{-1}e_{g-2}^{-1}f_{g-2}^{-1}\,(c_{2g-2}e_{g-2}c_{2g-2}f_{g-2})^2\,x_{g-1}c_{2g-1}e_{g-2}^{-1}f_{g-2}^{-1}c_{2g+1}^{-1}c_{2g+1}^{-1}\\ (c_{2g+1}c_{2g})^2\end{gathered} \tag{2.17}$$

and juxtaposing them results in

$$\begin{gathered}(c_1c_2)^2\,x_1c_3c_1^{-1}c_1^{-1}e_1^{-1}f_1^{-1}\,(c_4e_1c_4f_1)^2\,x_2c_5e_1^{-1}f_1^{-1}e_2^{-1}f_2^{-1}c_6e_2c_6f_2\\ x_3c_7e_2^{-1}f_2^{-1}e_3^{-1}f_3^{-1}\,(c_8e_3c_8f_3)^2\,x_4c_9e_3^{-1}f_3^{-1}e_4^{-1}f_4^{-1}c_{10}e_4c_{10}f_4\end{gathered}$$

$$x_5c_{11}e_4^{-1}f_4^{-1}e_5^{-1}f_5^{-1}\left(c_8e_3c_8f_3\right)^2 x_6c_{13}e_5^{-1}f_5^{-1}e_6^{-1}f_6^{-1}c_{14}e_6c_{14}f_6$$

$$\vdots \tag{2.18}$$

$$c_{2g-4}e_{g-3}c_{2g-4}f_{g-3}x_{g-2}c_{2g-3}e_{g-3}^{-1}f_{g-3}^{-1}e_{g-2}^{-1}f_{g-2}^{-1}c_{2g-2}e_{g-2}c_{2g-2}f_{g-2}$$

$$c_{2g-2}e_{g-2}c_{2g-2}f_{g-2}x_{g-1}c_{2g-1}e_{g-2}^{-1}f_{g-2}^{-1}c_{2g+1}^{-1}c_{2g+1}^{-1}c_{2g+1}c_{2g}c_{2g+1}c_{2g}$$

Eliminating the negative exponents using braid and commutativity relations in the first half of (2.18) results in

$$c_1dx_1c_3r_1e_1e_1c_4x_2c_5f_2^{-1}c_6x_3c_7r_3e_3e_3c_8x_4c_9f_4^{-1}c_{10}x_5c_{11}r_5e_5e_5c_{12}x_6c_{13}f_6^{-1}c_{14}x_7c_{15}, \tag{2.19}$$

where $d = c_1c_2c_1^{-1}$ and $r_i = f_i^{-1}c_{2i+2}f_i, i = 1, 3, 5$, just like in the even case (See (2.9) - and (2.11)).

The last part is also simplified using braid relation first

$$\underline{c_{2g-4}e_{g-3}c_{2g-4}}f_{g-3}x_{g-2}c_{2g-3}e_{g-3}^{-1}f_{g-3}^{-1}e_{g-2}^{-1}f_{g-2}^{-1}\underline{c_{2g-2}e_{g-2}c_{2g-2}}f_{g-2}$$

$$\underline{c_{2g-2}e_{g-2}c_{2g-2}}f_{g-2}x_{g-1}c_{2g-1}e_{g-2}^{-1}f_{g-2}^{-1}c_{2g+1}^{-1}c_{2g+1}^{-1}c_{2g+1}\underline{c_{2g}c_{2g+1}c_{2g}}$$

$$\downarrow$$

$$e_{g-3}c_{2g-4}e_{g-3}f_{g-3}x_{g-2}c_{2g-3}e_{g-3}^{-1}f_{g-3}^{-1}e_{g-2}^{-1}f_{g-2}^{-1}e_{g-2}c_{2g-2}e_{g-2}f_{g-2}$$

$$e_{g-2}c_{2g-2}e_{g-2}f_{g-2}x_{g-1}c_{2g-1}e_{g-2}^{-1}f_{g-2}^{-1}c_{2g+1}^{-1}c_{2g+1}^{-1}c_{2g+1}c_{2g+1}c_{2g}c_{2g+1}$$

and commutativity relation along with cancelation next

$$e_{g-3}c_{2g-4}\underline{e_{g-3}f_{g-3}}x_{g-2}c_{2g-3}\underline{e_{g-3}^{-1}f_{g-3}^{-1}}\ \underline{e_{g-2}^{-1}}f_{g-2}^{-1}\underline{e_{g-2}}c_{2g-2}e_{g-2}f_{g-2}$$

$$e_{g-2}c_{2g-2}\underline{e_{g-2}f_{g-2}}x_{g-1}c_{2g-1}\underline{e_{g-2}^{-1}f_{g-2}^{-1}}\ \underline{c_{2g+1}^{-1}c_{2g+1}^{-1}c_{2g+1}c_{2g+1}}c_{2g}c_{2g+1}$$

$$\downarrow$$

$$e_{g-3}c_{2g-4}x_{g-2}c_{2g-3}f_{g-2}^{-1}c_{2g-2}e_{g-2}f_{g-2}e_{g-2}c_{2g-2}x_{g-1}c_{2g-1}c_{2g}c_{2g+1}$$

Finally, defining $r_{g-2} = f_{g-2}^{-1}c_{2g-2}f_{g-2}$ and using commutativity one more time we obtain:

$$e_{g-3}c_{2g-4}x_{g-2}c_{2g-3}f_{g-2}^{-1}c_{2g-2}\underline{e_{g-2}f_{g-2}}e_{g-2}c_{2g-2}x_{g-1}c_{2g-1}c_{2g}c_{2g+1}$$

$$\downarrow$$

$$e_{g-3}c_{2g-4}x_{g-2}c_{2g-3}r_{g-2}f_{g-2}e_{g-2}c_{2g-2}x_{g-1}c_{2g-1}c_{2g}c_{2g+1}$$

Putting the two ends together, the word now has the form

$$c_1dx_1c_3r_1e_1e_1c_4x_2c_5f_2^{-1}c_6x_3c_7r_3e_3e_3c_8x_4c_9f_4^{-1}c_{10}x_5c_{11}r_5e_5e_5c_{12}x_6c_{13}f_6^{-1}c_{14}x_7c_{15},$$

$$\vdots \tag{2.20}$$

$$c_{2g-4}x_{g-2}c_{2g-3}r_{g-2}f_{g-2}e_{g-2}c_{2g-2}x_{g-1}c_{2g-1}c_{2g}c_{2g+1}$$

In a more compact form we have

$$c_1dx_1c_3r_1e_1e_1c_4x_2c_5f_2^{-1}W_6W_8\cdots W_{g-1}W_g,$$

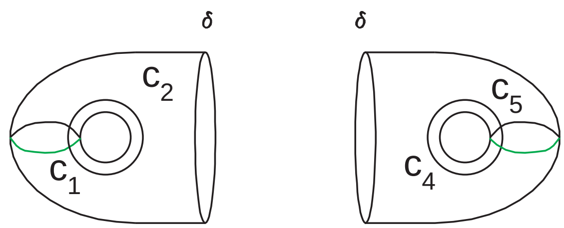
where $W_i = c_{2i-6}x_{i-3}c_{2i-5}r_{i-3}e_{i-3}e_{i-3}c_{2i-4}x_{i-2}c_{2i-3}f_{i-2}^{-1}$ and $W_g = c_{2g-6}x_{g-3}c_{2g-5}r_{g-3}e_{g-3}e_{g-3}c_{2g-4}x_{g-2}c_{2g-3}r_{g-2}f_{g-2}e_{g-2}c_{2g-2}x_{g-1}c_{2g-1}c_{2g}c_{2g+1}$.

The next section deals with lower genus.

3. Low genus

For genus 2 and 3 there isn't much difficulty with eliminating the terms with negative exponents. For genus 4, 5, and 6 however, we use additional lantern relations to eliminate them.

3.1. **genus 2.** We glue two tori with one one boundary component together and juxtapose the words $(c_1c_2)^2$ and $(c_5c_4)^{-2}$ on the resulting closed surface.



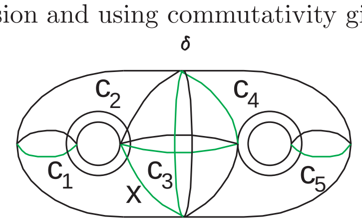
Next, we use the first relation in (1.2) to replace $(c_5c_4)^{-2}$ by $\delta^{-1}(c_5c_4)^4$. Now using the lantern relation

$$\delta x c_3 = c_1^2 c_5^2$$

we substitute $\delta^{-1} = xc_3c_1^{-2}c_5^{-2}$ and obtain

$$(c_1c_2)^2(c_5c_4)^{-2} = (c_1c_2)^2\delta^{-1}(c_5c_4)^4 = (c_1c_2)^2 xc_3c_1^{-2}c_5^{-2}(c_5c_4)^4.$$

Expanding the expression and using commutativity gives



$$c_1c_2c_1c_2c_1^{-2}xc_3c_5^{-2}c_5c_4c_5c_4c_5c_4c_5c_4 = c_1\underline{c_2c_1c_2}c_1^{-2}xc_3c_5^{-1}\underline{c_4c_5c_4}c_5c_4c_5c_4.$$

Using braid relation on the underlined terms and doing the obvious cancelations afterward we arrive at

$$c_1c_1c_2c_1c_1^{-2}xc_3c_5^{-1}c_5c_4c_5c_5c_4c_5c_4 = c_1c_1c_2c_1^{-1}xc_3c_4c_5c_5c_4c_5c_4.$$

Even though $c_1c_2c_1^{-1}$ represents a positive twist, we can do away with this conjugation with little effort: Just bring the left most c_1 in the third power of the word to the right end and see the cancelations that occur between the underlined terms as you go from right to left.

$$\begin{aligned}
&\left(c_1c_1c_2c_1^{-1}xc_3c_4c_5c_5c_4c_5c_4\right)^3\\
&= \underline{c_1}c_1c_2c_1^{-1}xc_3c_4c_5c_5c_4c_5c_4c_1c_1c_2c_1^{-1}xc_3c_4c_5c_5c_4c_5c_4c_1c_1c_2c_1^{-1}xc_3c_4c_5c_5c_4c_5c_4\\
&\equiv c_1c_2\underline{c_1^{-1}}xc_3c_4c_5c_5c_4c_5c_4\underline{c_1}c_1c_2\underline{c_1^{-1}}xc_3c_4c_5c_5c_4c_5c_4\underline{c_1}c_1c_2\underline{c_1^{-1}}xc_3c_4c_5c_5c_4c_5c_4\underline{c_1}\\
&= c_1c_2xc_3c_4c_5c_5c_4c_5c_4\ c_1c_2xc_3c_4c_5c_5c_4c_5c_4\ c_1c_2xc_3c_4c_5c_5c_4c_5c_4
\end{aligned}$$

$$(3.1) = \left(c_1c_2xc_3c_4c_5c_5c_4c_5c_4\right)^3 = 1$$

3.1.1. *An Alternate Expression.* We can obtain an alternate expression out of (3.1) by inserting into it lantern relations as follows:

$$
\begin{aligned}
& \left(c_1c_2xc_3c_4c_5c_5c_4c_5c_4\right)^3 \\
\equiv\ & \left(c_1c_2xc_3c_4c_3^{-2}c_3^2c_5^2c_4c_5c_4\right)^3 \\
\equiv\ & \left(c_1c_2xc_3c_4c_3^{-2}k_1h_1c_1c_4c_5c_4\right)^3 \\
\equiv\ & \left(c_2xc_3c_4c_3^{-2}k_1h_1c_1^2c_5^2c_5^{-2}c_4c_5c_4\right)^3 \\
\equiv\ & \left(c_2xc_3c_4c_3^{-2}k_1h_1c_3\delta xc_5^{-2}c_4c_5c_4\right)^3 \\
\equiv\ & \left(c_2xc_3c_4c_3^{-1}k_1h_1\delta xc_5^{-2}c_5c_4c_5\right)^3 \\
\equiv\ & \left(c_2x\left(c_3c_4c_3^{-1}\right)k_1h_1\delta x\left(c_5^{-1}c_4c_5\right)\right)^3 \\
\equiv\ & \left(c_2xt_2k_1h_1\delta xs_2\right)^3 = 1
\end{aligned}
\tag{3.2}
$$

where $t_2 = c_3c_4c_3^{-1}$ and $s_2 = c_5^{-1}c_4c_5$. The cycles that are used in the first lantern relation

$$c_1k_1h_1 \quad = \quad c_3^2c_5^2$$

used in (3.2) are shown in Figure 4.

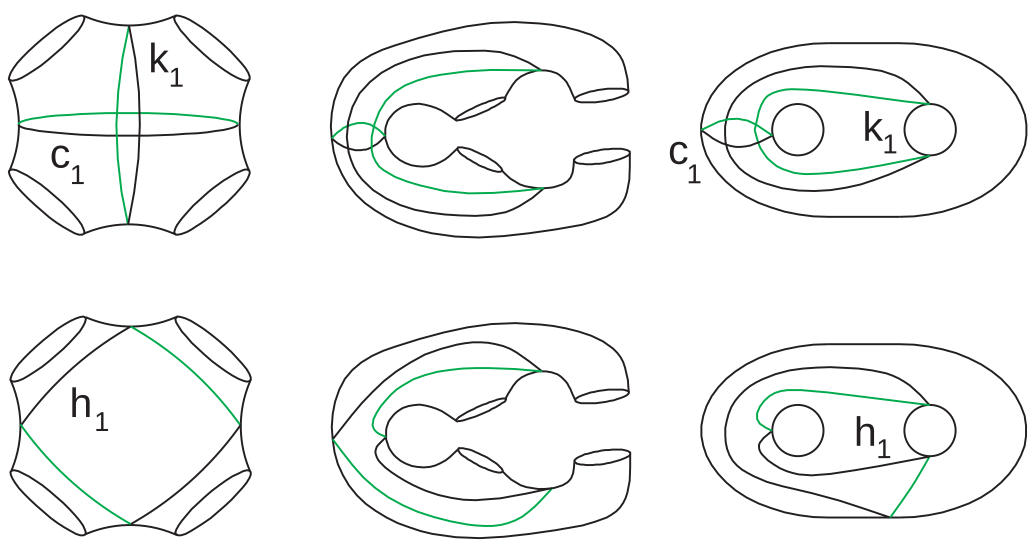


FIGURE 4.

3.2. **genus 3.** To get the word for genus 3 we will use (2.17) with $g = 3$:

$$
\begin{array}{r}
\left(c_1c_2\right)^2 \\
x_1c_3c_1^{-1}c_1^{-1}e_1^{-1}f_1^{-1}\left(c_4e_1c_4f_1\right)^2 x_2c_5e_1^{-1}f_1^{-1}c_7^{-1}c_7^{-1} \\
\left(c_7c_6\right)^2,
\end{array}
$$

i.e.

$$c_1c_2c_1c_2\ x_1c_3c_1^{-1}c_1^{-1}e_1^{-1}f_1^{-1}c_4e_1c_4f_1c_4e_1c_4\underline{f_1}x_2c_5e_1^{-1}\underline{f_1^{-1}}c_7^{-1}\underline{c_7^{-1}}\ \underline{c_7}c_6c_7c_6.$$

After this initial cancelation we use braid relation on the underlined terms

$$c_1\underline{c_2c_1c_2}\ x_1c_3c_1^{-1}c_1^{-1}e_1^{-1}f_1^{-1}\underline{c_4e_1c_4}f_1\underline{c_4e_1c_4}x_2c_5e_1^{-1}c_7^{-1}\underline{c_6c_7c_6},$$

and get

$$c_1c_1c_2\underline{c_1}\ x_1c_3\underline{c_1^{-1}}c_1^{-1}\underline{e_1^{-1}}f_1^{-1}\underline{e_1}c_4e_1f_1e_1c_4\underline{e_1}x_2c_5\underline{e_1^{-1}c_7^{-1}c_7}c_6c_7.$$

Cancelation of the underlined terms gives

$$c_1c_1c_2\ x_1c_3c_1^{-1}f_1^{-1}c_4e_1f_1e_1c_4x_2c_5c_6c_7. \tag{3.3}$$

Now, rearranging the terms using commutativity and letting $r = f_1^{-1}c_4f_1$ we obtain

$$c_1c_1c_2c_1^{-1}x_1c_3re_1e_1c_4x_2c_5c_6c_7.$$

Using the same kind of rotation as in (3.1) will allow us to eliminate the conjugation $c_1c_2c_1^{-1}$ and we will get

$$(c_1c_2x_1c_3rc_8c_8c_4x_2c_5c_6c_7)^3 = 1 \tag{3.4}$$

in the end, using the identification $e_1 = c_8$ as shown in Figure 3.

3.2.1. *An Alternate Expression.* An alternate expression is obtained when f_1^{-1} is eliminated from (3.3) using the lantern relation

$$f_1tv = c_1c_3c_5c_7.$$

Substituting $f_1^{-1} = tvc_1^{-1}c_3^{-1}c_5^{-1}c_7^{-1}$ into (3.3) we get

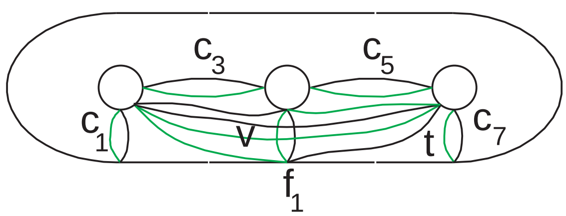


Figure 5.

$$c_1c_1c_2x_1\underline{c_3}c_1^{-1}tvc_1^{-1}\underline{c_3^{-1}}c_5^{-1}c_7^{-1}c_4e_1f_1e_1c_4x_2c_5c_6c_7.$$

We can cancel the underlined terms and rewrite the rest of the word as

$$c_1c_1c_2c_1^{-1}c_1^{-1}x_1tvc_5^{-1}c_4e_1f_1e_1c_4x_2c_5c_7^{-1}c_6c_7$$

using commutativity. Now, because $c_5(\bar{x}_2) = x_2$ (i.e., Dehn twist about c_5 maps $\bar{x}_2$ to x_2) we have

$$c_5\bar{x}_2c_5^{-1} = x_2, \text{ i.e., } c_5\bar{x}_2 = x_2c_5.$$

Substituting $c_5\bar{x}_2$ in place of x_2c_5 and inserting a $c_5c_5^{-1}$ using commutativity results in

$$c_1c_1c_2c_1^{-1}c_1^{-1}x_1tvc_5^{-1}c_4c_5e_1f_1e_1c_5^{-1}c_4c_5\bar{x}_2c_7^{-1}c_6c_7.$$

Now, all we have to do is rename the conjugations. If we let

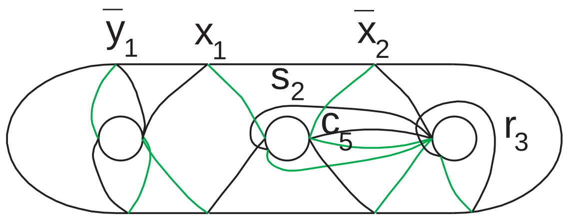


FIGURE 6.

$$\bar{y}_1 = c_1 c_1 c_2 c_1^{-1} c_1^{-1}, s_2 = c_5^{-1} c_4 c_5, \quad \text{and} \quad r_3 = c_7^{-1} c_6 c_7$$

then the final form of the word becomes

$$(\bar{y}_1 x_1 t v s_2 c_8 f_1 c_8 s_2 \bar{x}_2 r_3)^3 = 1, \tag{3.5}$$

using the identification $e_1 = c_8$ in Figure 3.

3.2.2. *Another Alternate Expression.* One other expression is obtained using the relation

$$(c_1 c_2 c_3)^4 = e_1 f_1$$

in order to substitute $(c_1 c_2 c_3)^4$ in place of $c_8 f_1 = e_1 f_1$. The alternate expression is

$$\left(\bar{y}_1 x_1 t v s_2 \,(c_1 c_2 c_3)^4\, c_8 s_2 \bar{x}_2 r_3\right)^3 = 1, \tag{3.6}$$

3.3. **genus 4.** Using (2.8) with $g = 4$ we have

$$\begin{gathered} (c_1 c_2)^2 \\ x_1 c_3 c_1^{-1} c_1^{-1} e_1^{-1} f_1^{-1} (c_4 e_1 c_4 f_1)^2 \, x_2 c_5 e_1^{-1} f_1^{-1} e_2^{-1} f_2^{-1} \\ c_6 e_2 c_6 f_2 \\ x_3 c_7 e_2^{-1} f_2^{-1} c_9^{-1} c_9^{-1} (c_9 c_8)^4, \end{gathered}$$

which will be the same as the top line in (2.10) for the most part after juxtaposing:

$$c_1 c_1 c_2 \underline{c_1} x_1 c_3 \underline{c_1}^{-1} c_1^{-1} \underline{e_1}^{-1} f_1^{-1} \underline{e_1} c_4 e_1 f_1 e_1 c_4 \underline{e_1 f_1} x_2 c_5 \underline{e_1^{-1} f_1}^{-1} \underline{e_2}^{-1} f_2^{-1} \underline{e_2} c_6 \underline{e_2 f_2} x_3 c_7 \; \underline{e_2^{-1} f_2^{-1}} c_9^{-1} \underline{c_9^{-1}} \left(\underline{c_9} c_8\right)^4.$$

Cancelation of the underlined terms gives

$$c_1 c_1 c_2 x_1 c_3 c_1^{-1} f_1^{-1} c_4 e_1 f_1 e_1 c_4 x_2 c_5 f_2^{-1} c_6 x_3 c_7 \quad c_9^{-1} \underline{c_8 c_9 c_8} c_9 c_8 c_9 c_8.$$

Rearranging some commuting terms along with braid relation on the underlined triple we get

$$c_1 c_1 c_2 c_1^{-1} x_1 c_3 f_1^{-1} c_4 f_1 e_1 e_1 c_4 x_2 c_5 f_2^{-1} c_6 x_3 c_7 \underline{c_9^{-1} c_9} c_8 c_9 c_9 c_8 c_9 c_8,$$

and by setting $d = c_1 c_2 c_1^{-1}, r_1 = f_1^{-1} c_4 f_1$ and canceling the underlined pair we arrive at

$$c_1 d x_1 c_3 r_1 e_1 e_1 c_4 x_2 c_5 f_2^{-1} c_6 x_3 c_7 c_8 c_9 c_9 c_8 c_9 c_8. \tag{3.7}$$

We will have to use another lantern relation to eliminate f_2^{-1} and that will be

$$f_2 t v = f_1 c_5 c_7 c_9$$

as shown in Figure 7. We solve it for f_2^{-1}

$$f_2^{-1} = t v f_1^{-1} c_5^{-1} c_7^{-1} c_9^{-1}$$

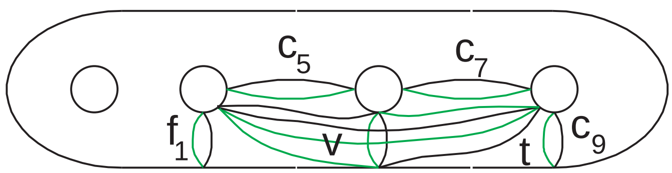


FIGURE 7.

and substituting that in (3.7) using commutativity gives :

$$c_1 d x_1 c_3 r_1 e_1 e_1 c_4 f_1^{-1} x_2 \underline{c_5} t v \underline{c_5}^{-1} c_7^{-1} c_6 x_3 c_7 c_9^{-1} c_8 c_9 \underline{c_9 c_8 c_9} c_8.$$

Because $c_7(\bar{x}_3) = x_3$ (i.e., Dehn twist about c_7 maps $\bar{x}_3$ to x_3) we have

$$c_7 \bar{x}_3 c_7^{-1} = x_3, \text{ i.e., } c_7 \bar{x}_3 = x_3 c_7. \tag{3.8}$$

Substituting that in and using braid relation and cancelation on the underlined parts we get

$$c_1 d x_1 c_3 r_1 e_1 e_1 c_4 f_1^{-1} x_2 t v c_7^{-1} c_6 c_7 \overline{x}_3 c_9^{-1} \underline{c_8 c_9 c_8} c_9 c_8 c_8.$$

Renaming $c_7^{-1} c_6 c_7 = s_3$ and using braid relation on the underlined part again gives

$$c_1 d x_1 c_3 \underline{r_1 e_1 e_1 c_4 f_1}^{-1} x_2 t v s_3 \overline{x}_3 \underline{c_9^{-1} c_9} c_8 c_9 c_9 c_8 c_8. \tag{3.9}$$

The following is how we eliminate f_1^{-1} from the underlined portion:

$$\begin{aligned} \underline{r_1} e_1 e_1 c_4 f_1^{-1} = f_1^{-1} c_4 f_1 e_1 e_1 c_4 f_1^{-1} &= \\ f_1^{-1} c_4 e_1 e_1 \underline{f_1 c_4 f_1^{-1}} &= \\ f_1^{-1} c_4 e_1 e_1 c_4^{-1} f_1 c_4 &= \\ f_1^{-1} c_4 e_1 \underline{c_4^{-1} f_1 f_1^{-1} c_4} e_1 c_4^{-1} f_1 c_4 &= \end{aligned}$$

$$f_1^{-1} c_4 e_1 c_4^{-1} f_1 f_1^{-1} c_4 e_1 c_4^{-1} f_1 c_4 = y_2 y_2 c_4, \tag{3.10}$$

where $y_2 = f_1^{-1} c_4 e_1 c_4^{-1} f_1$. Now (3.9) becomes

$$c_1 d x_1 c_3 y_2 y_2 c_4 x_2 t v s_3 \overline{x}_3 c_8 c_9 c_9 c_8 c_8.$$

Using the same rotation operation as in (3.1) allows us to eliminate the conjugation $d = c_1 c_2 c_1^{-1}$ and we get

$$(c_1 c_2 x_1 c_3 y_2 y_2 c_4 x_2 t v s_3 \overline{x}_3 c_8 c_9 c_9 c_8 c_8)^3 = 1. \tag{3.11}$$

3.3.1. *An Alternate Expression.* An alternate expression is obtained when f_1^{-1} is eliminated from (3.9) using the lantern relation

$$f_1 t_{1,4} v_{1,4} = c_1 c_3 v c_9.$$

Substituting $f_1^{-1} = t_{1,4} v_{1,4} c_1^{-1} c_3^{-1} v^{-1} c_9^{-1}$ into (3.9) yields

$$c_1 d x_1 c_3 r_1 e_1 e_1 c_4 t_{1,4} v_{1,4} c_1^{-1} c_3^{-1} v^{-1} c_9^{-1} x_2 t v s_3 \overline{x}_3 c_8 c_9 c_9 c_8 c_8. \tag{3.12}$$

Using commutativity and inserting identity where necessary we can rewrite the last expression as

$$c_1 c_1 c_2 c_1^{-1} c_1^{-1} x_1 c_3 r_1 c_3^{-1} e_1 e_1 c_3 c_4 c_3^{-1} t_{1,4} v_{1,4} v^{-1} x_2 v v^{-1} t v s_3 \overline{x}_3 c_9^{-1} c_8 c_9 c_9 c_8 c_8, \tag{3.13}$$

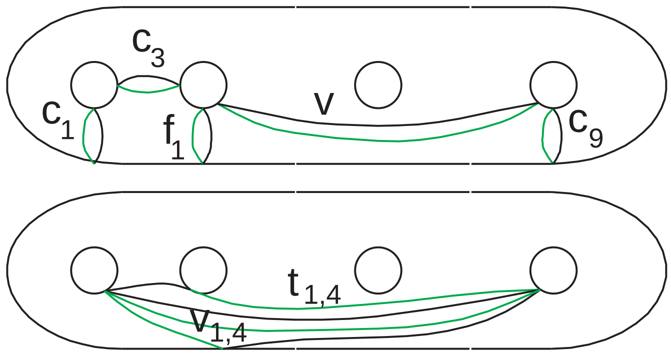


FIGURE 8.

remembering $d = c_1c_2c_1^{-1}$. Now, all we have to do is rename the conjugations:

$\bar{y}_1 = c_1c_1c_2c_1^{-1}c_1^{-1}, u_1 = c_3r_1c_3^{-1}, \bar{s_2} = c_3c_4c_3^{-1}, w = v^{-1}x_2v, z = v^{-1}tv$ and $r_4 = c_9^{-1}c_8c_9$.

Then (3.13) becomes $\bar{y}_1x_1u_1e_1e_1\bar{s_2}t_{1,4}v_{1,4}wzs_3\overline{x}_3r_4c_9c_8c_8$. Therefore the final form of the alternate word for genus 4 is

$$(\bar{y}_1x_1u_1e_1e_1\bar{s_2}t_{1,4}v_{1,4}wzs_3\overline{x}_3r_4c_9c_8c_8)^3 = 1. \tag{3.14}$$

3.3.2. *An alternate Construction.* An alternate gluing operation for genus 4 can be performed as shown in Figure 9.

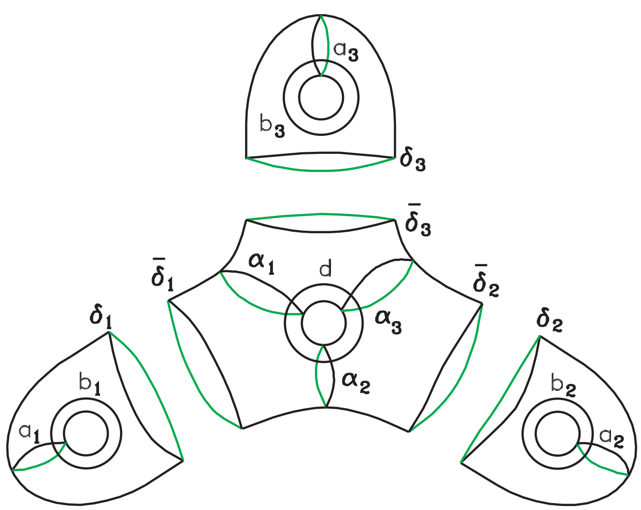


FIGURE 9.

We use the words

$$b_1a_1b_1a_1$$
$$b_2a_2b_2a_2$$
$$b_3a_3b_3a_3 \tag{3.15}$$
$$(\alpha_1\alpha_2\alpha_3d)^{-1}$$

on the four bounded surfaces taking the one in the center with the opposite orientation. Using the star relation (1.2)

$$(\alpha_1\alpha_2\alpha_3d)^3 = \bar{\delta}_1\bar{\delta}_2\bar{\delta}_3,$$

we write

$$(\alpha_1\alpha_2\alpha_3d)^{-1} = \bar{\delta}_1^{-1}\bar{\delta}_2^{-1}\bar{\delta}_3^{-1}(\alpha_1\alpha_2\alpha_3d)^2$$

and using the lantern relations

$$\delta_1x_1c_1 = a_1a_1\alpha_1\alpha_2$$
$$\delta_2x_2c_2 = a_2a_2\alpha_2\alpha_3$$
$$\delta_3x_3c_3 = a_3a_3\alpha_3\alpha_1$$

and the fact that $\delta_1 = \bar{\delta}_1, \delta_2 = \bar{\delta}_2, \delta_3 = \bar{\delta}_3$ we write

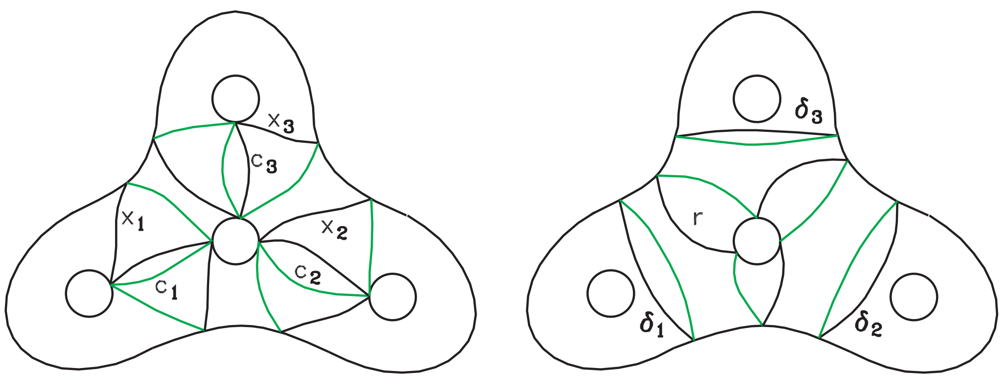


FIGURE 10.

$$\delta_1^{-1} = x_1c_1a_1^{-1}a_1^{-1}\alpha_1^{-1}\alpha_2^{-1}$$
$$\delta_2^{-1} = x_2c_2a_2^{-1}a_2^{-1}\alpha_2^{-1}\alpha_3^{-1}$$
$$\delta_3^{-1} = x_3c_3a_3^{-1}a_3^{-1}\alpha_3^{-1}\alpha_1^{-1}$$

Substituting all these in (3.15) and juxtaposing we obtain

$$b_1a_1b_1\underline{a_1}b_2a_2b_2\underline{a_2}b_3a_3b_3\underline{a_3}x_1c_1\underline{a_1^{-1}}a_1^{-1}\alpha_1^{-1}\alpha_2^{-1}x_2c_2\underline{a_2^{-1}}a_2^{-1}\alpha_2^{-1}\alpha_3^{-1}x_3c_3\underline{a_3^{-1}}a_3^{-1}\alpha_3^{-1}\alpha_1^{-1}(\alpha_1\alpha_2\alpha_3d)^2$$

We can cancel the underlined terms right away using commutativity and rearrange rest of the word as

$$\underline{b_1a_1b_1}a_1^{-1}\underline{b_2a_2b_2}a_2^{-1}\underline{b_3a_3b_3}a_3^{-1}x_1c_1x_2c_2x_3c_3\alpha_1^{-1}\alpha_2^{-1}\underline{\alpha_2^{-1}}\alpha_3^{-1}\underline{\alpha_3^{-1}}\ \underline{\alpha_1^{-1}}\ \underline{\alpha_1}\ \underline{\alpha_2}\ \underline{\alpha_3}d\alpha_1\alpha_2\alpha_3d$$

using commutativity again. Now, using braid relation and cancelation on the underlined portion, the word reduces to

$$a_1b_1\underline{a_1}\ \underline{a_1^{-1}}a_2b_2\underline{a_2}\ \underline{a_2^{-1}}a_3b_3\underline{a_3}\ \underline{a_3^{-1}}x_1c_1x_2c_2x_3c_3\alpha_1^{-1}\alpha_2^{-1}\alpha_3^{-1}d\alpha_1\alpha_2\alpha_3d$$

Further cancelation and renaming $r=(\alpha_1\alpha_2\alpha_3)^{-1}d\alpha_1\alpha_2\alpha_3$ gives the positive relator

$$(a_1b_1a_2b_2a_3b_3x_1c_1x_2c_2x_3c_3rd)^3=1 \tag{3.16}$$

3.3.3. *Another alternate expression.* We can modify (3.16) in order to insert the lantern relation

$$\alpha_2tv=a_1c_1a_2c_2$$

into it and obtain a new expression.

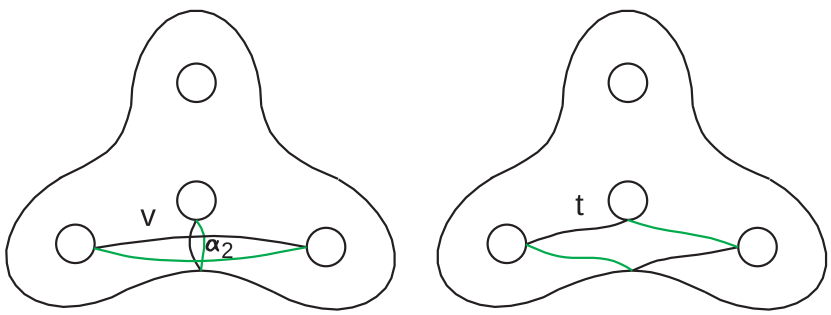


Figure 11.

$$\begin{aligned}&\left(a_1b_1a_1^{-1}a_2b_2a_2^{-1}a_3b_3x_1x_2a_1a_2c_1c_2x_3c_3rd\right)^3\\ =\ &(g_1g_2a_3b_3x_1x_2\alpha_2tvx_3c_3rd)^3=1,\end{aligned} \tag{3.17}$$

where $g_1=a_1b_1a_1^{-1}$ and $g_2=a_2b_2a_2^{-1}$.

3.4. **genus 5.** Using (2.17) with $g=5$ we have

$$\begin{aligned}&(c_1c_2)^2\\ x_1c_3c_1^{-1}c_1^{-1}e_1^{-1}f_1^{-1}(c_4e_1c_4f_1)^2\,x_2c_5e_1^{-1}f_1^{-1}e_2^{-1}f_2^{-1}&\\ &c_6e_2c_6f_2\\ x_3c_7e_2^{-1}f_2^{-1}e_3^{-1}f_3^{-1}(c_8e_3c_8f_3)^2\,x_4c_9e_3^{-1}f_3^{-1}c_{11}^{-1}c_{11}^{-1}&\\ &c_{11}c_{10}c_{11}c_{10}\end{aligned}$$

The top two lines in (2.18) up to c_{10} becomes

$$c_1dx_1c_3r_1e_1e_1c_4x_2c_5f_2^{-1}c_6x_3c_7r_3e_3e_3c_8x_4c_9c_{11}^{-1}c_{11}^{-1}$$

in (2.19), and this followed by

$$c_{11}c_{10}c_{11}c_{10}=c_{11}c_{11}c_{10}c_{11}$$

gives

$$c_1dx_1c_3r_1e_1e_1c_4x_2c_5f_2^{-1}c_6x_3c_7r_3e_3e_3c_8x_4c_9c_{11}^{-1}c_{11}^{-1}c_{11}c_{11}c_{10}c_{11}=$$

$$c_1 d x_1 c_3 r_1 e_1 e_1 c_4 x_2 c_5 f_2^{-1} c_6 x_3 c_7 r_3 e_3 e_3 c_8 x_4 c_9 c_{10} c_{11} \tag{3.18}$$

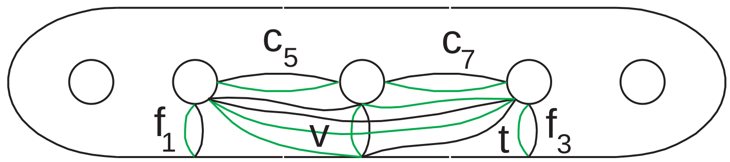


FIGURE 12.

We will use the same additional lantern relation as in genus 4 to eliminate f_2^{-1}: Substitute

$$f_2^{-1} = tvf_1^{-1}c_5^{-1}c_7^{-1}f_3^{-1}$$

in (3.18) using commutativity

$$c_1 d x_1 c_3 r_1 e_1 e_1 c_4 f_1^{-1} x_2 c_5 tv c_5^{-1} c_7^{-1} c_6 x_3 c_7 f_3^{-1} r_3 e_3 e_3 c_8 x_4 c_9 c_{10} c_{11}$$

and eliminate f_1^{-1}, c_5^{-1}, and c_7^{-1} in the exact same way as in genus 4 following Figure 7. Therefore we can borrow the portion of (3.11) up to c_8 and write

$$c_1 d x_1 c_3 y_2 y_2 c_4 x_2 tv s_3 \overline{x}_3 \underline{f_3^{-1} r_3 e_3 e_3 c_8} x_4 c_9 c_{10} c_{11}. \tag{3.19}$$

The following is how we deal with f_3^{-1}:

$$f_3^{-1} r_3 e_3 e_3 c_8 = f_3^{-1} \underline{f_3^{-1} c_8 f_3} e_3 e_3 c_8 =$$

$$f_3^{-1} c_8 f_3 c_8^{-1} e_3 e_3 c_8 = f_3^{-1} c_8 f_3 c_8^{-1} e_3 c_8 c_8^{-1} e_3 c_8 = r_3 \overline{r}_3 \overline{r}_3,$$

where $r_3 = f_3^{-1} c_8 f_3, \overline{r}_3 = c_8^{-1} e_3 c_8$. Therefore the final form of the genus 5 word is

$$\left(c_1 c_2 x_1 c_3 y_2 y_2 c_4 x_2 tv s_3 \overline{x}_3 r_3 \overline{r}_3 \overline{r}_3 x_4 c_9 c_{10} c_{11}\right)^3 = 1 \tag{3.20}$$

after a rotation similar to (3.1) applied.

3.5. **genus 6.** Setting $g = 6$ in (2.8) we obtain the components

$$
\begin{gathered}
(c_1 c_2)^2 \\
x_1 c_3 c_1^{-1} c_1^{-1} e_1^{-1} f_1^{-1} \left(c_4 e_1 c_4 f_1\right)^2 x_2 c_5 e_1^{-1} f_1^{-1} e_2^{-1} f_2^{-1} \\
c_6 e_2 c_6 f_2 \\
x_3 c_7 e_2^{-1} f_2^{-1} e_3^{-1} f_3^{-1} \left(c_8 e_3 c_8 f_3\right)^2 x_4 c_9 e_3^{-1} f_3^{-1} e_4^{-1} f_4^{-1} \\
c_{10} e_4 c_{10} f_4 \\
x_5 c_{11} e_4^{-1} f_4^{-1} c_{13}^{-1} c_{13}^{-1} \left(c_{13} c_{12}\right)^4
\end{gathered} \tag{3.21}
$$

of the word on bounded subsurfaces before juxtaposition. After juxtaposing them we arrive at

$$c_1 d x_1 c_3 r_1 e_1 e_1 c_4 x_2 c_5 f_2^{-1} c_6 x_3 c_7 r_3 e_3 e_3 c_8 x_4 c_9 f_4^{-1} c_{10} x_5 c_{11} c_{12} c_{13} c_{13} c_{12} c_{13} c_{12}$$

as in (2.13) with $g = 6$. If we substitute

$$f_2^{-1} = t_{2,4} v_{2,4} f_1^{-1} c_5^{-1} c_7^{-1} f_3^{-1} \quad \text{and} \quad f_4^{-1} = t_{4,6} v_{4,6} f_3^{-1} c_9^{-1} c_{11}^{-1} c_{13}^{-1}$$

then we get

$$c_1 d x_1 c_3 r_1 e_1 e_1 c_4 x_2 \underline{c_5} t_{2,4} v_{2,4} f_1^{-1} \underline{c_5}^{-1} c_7^{-1} f_3^{-1} c_6 x_3 c_7 r_3 e_3 e_3 c_8 x_4 \underline{c_9} t_{4,6} v_{4,6} f_3^{-1} \underline{c_9}^{-1} c_{11}^{-1} c_{13}^{-1} c_{10} x_5 c_{11} c_{12} c_{13} c_{13} c_{12} c_{13} c_{12}.$$

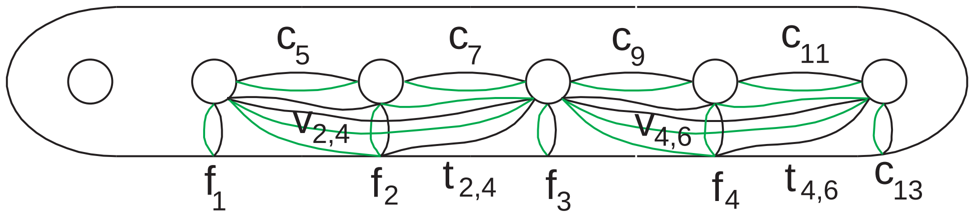


FIGURE 13.

We do the obvious cancelations and use (3.8) again to write

$$x_3c_7 = c_7\bar{x}_3 \text{ and likewise } x_5c_{11} = c_{11}\bar{x}_5.$$

Using commutativity as well yields

$$c_1dx_1c_3\underline{r_1e_1e_1c_4f_1^{-1}}x_2t_{2,4}v_{2,4}c_7^{-1}c_6c_7\bar{x}_3f_3^{-1}\underline{r_3e_3e_3c_8f_3^{-1}}x_4t_{4,6}v_{4,6}c_{11}^{-1}c_{10}c_{11}\bar{x}_5c_{13}^{-1}c_{12}c_{13}c_{13}c_{12}c_{13}c_{12}.$$

Following the same argument given in (3.10) for the underlined portions and renaming $c_7^{-1}c_6c_7 = s_3, c_{11}^{-1}c_{10}c_{11} = s_5$ and $c_{13}^{-1}c_{12}c_{13} = s_6$ we get

$$c_1dx_1c_3y_2y_2c_4x_2t_{2,4}v_{2,4}s_3\bar{x}_3f_3^{-1}y_4y_4c_8x_4t_{4,6}v_{4,6}s_5\bar{x}_5s_6c_{13}c_{12}c_{13}c_{12}.$$

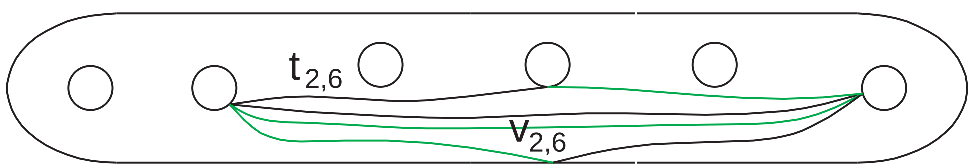


FIGURE 14.

One more lantern substitution is needed to eliminate f_3 and that is

$$f_3^{-1} = t_{2,6}v_{2,6}f_1^{-1}v_{2,4}^{-1}v_{4,6}^{-1}c_{13}^{-1}.$$

Result from substituting that is

$$c_1dx_1c_3y_2y_2c_4x_2t_{2,4}v_{2,4}s_3\bar{x}_3t_{2,6}v_{2,6}f_1^{-1}v_{2,4}^{-1}v_{4,6}^{-1}c_{13}^{-1}y_4y_4c_8x_4t_{4,6}v_{4,6}s_5\bar{x}_5s_6c_{13}c_{12}c_{13}c_{12}.$$

The idea in (3.8) can be used for $t_{4,6}$ and $v_{4,6}$ to write

$$t_{4,6}v_{4,6} = v_{4,6}\bar{t}_{4,6}.$$

Now, conjugating y_4, c_8, x_4 by $v_{4,6}$ and renaming them as $y_{4,6}c_{8,12}x_{4,5}$, respectively, and likewise renaming $v_{2,4}\bar{x}_3v_{2,4}^{-1} = \bar{x}_{3,2}$ and $y_6 = c_{13}^{-1}s_6c_{13}$ gives

$$c_1dx_1c_3y_2y_2c_4x_2t_{2,4}s_3\bar{x}_{3,2}t_{2,6}v_{2,6}f_1^{-1}y_{4,6}y_{4,6}c_{8,12}x_{4,5}\bar{t}_{4,6}s_5\bar{x}_5y_6c_{12}c_{13}c_{12}.$$

The final lantern substitution is to replace f_1^{-1} and it is

$$f_1^{-1} = t_{1,6}v_{1,6}c_1^{-1}c_3^{-1}v_{2,6}^{-1}c_{13}^{-1},$$

where $t_{1,6}$ and $v_{1,6}$ are defined in the same fashion. Substituting that results in

$$c_1dx_1c_3y_2y_2c_4x_2t_{2,4}s_3\overline{x}_{3,2}t_{2,6}v_{2,6}t_{1,6}v_{1,6}c_1^{-1}c_3^{-1}v_{2,6}^{-1}c_{13}^{-1}y_{4,6}y_{4,6}c_{8,12}x_{4,5}\bar{t}_{4,6}s_5\overline{x}_5y_6c_{12}c_{13}c_{12}.$$

After renaming two conjugations $t_{1,2,6} = v_{2,6}t_{1,6}v_{2,6}^{-1}$ and $w_6 = c_{13}^{-1}y_6c_{13}$ following the braid relation $c_{12}c_{13}c_{12} = c_{13}c_{12}c_{13}$ we will push c_1^{-1} to the right end in order to cancel it with the c_1 at the left end of the next copy. Following the same idea for c_3^{-1} gives $u_1 = c_3^{-1}dc_3$ after we invoke (3.8) one more time in order to write $x_1c_3 = c_3\bar{x}_1$. All of these changes are realized in the final form of the genus 6 word that follows:

(3.22)
$$\left(u_1\bar{x}_1y_2y_2c_4x_2t_{2,4}s_3\bar{x}_{3,2}t_{2,6}t_{1,2,6}v_{1,6}y_{4,6}y_{4,6}c_{8,12}x_{4,5}\bar{t}_{4,6}s_5\bar{x}_5w_6c_{12}c_{13}\right)^3 = 1.$$

4. Applications

In this section we will compute the homeomorphism invariants of the 4-manifolds defined by the words in the previous section. We will denote by X_g the manifolds that are given by the words (3.1), (3.4), and (3.16) and those that are obtained from them by inserting k lantern relations will be denoted by $X_{g,k}$.

Proposition 4.1. *The signature and Euler characteristic of the Lefschetz fibration $X_g, g = 2, 3, 4,$ is given by $\sigma(X_g) = -2\,(g+7)$ and $\chi\,(X_g) = 2g + 22$, respectively.*

Proof: By checking the respective equations we see that the number of cycles in those that define X_2, X_3, and X_4 is $3\,(2g+6)$; therefore their Euler characteristics are given by the formula

$$\chi\,(X_g) = 2\,(2-2g) + 3\,(2g+6) = 2g + 22.$$

Here we used the well known fact from the theory of Lefschetz fibrations that the Euler characteristic of a Lefschetz fibration $X^4 \to S^2$ is given by the formula

$$\chi(X) = 4 - 4g + s, \tag{4.1}$$

where g is the genus of the fiber and s is the number of singular fibers, i.e., the number of vanishing cycles [4].

For signature computations that follow the reader is referred to article [1]. First we compute $\sigma(X_2)$.

Let C_2 denote a chain of length 2 in $\mathcal{M}_2$, such as $(c_1c_2)^6\,\delta^{-1}$ and $(c_5c_4)^6\,\delta^{-1}$. Following the construction of the word in 3.1 we have

$$\begin{aligned}
& C_2 \cdot C_2^{-1} \\
= \;& (c_1c_2)^6\,\delta^{-1}\delta\,(c_5c_4)^{-6} = (c_1c_2)^6\,(c_5c_4)^{-6} \\
\equiv \;& \left((c_1c_2)^2\,(c_5c_4)^{-2}\right)^3 \quad \text{(commutativity)} \\
\equiv \;& \left((c_1c_2)^2\,\delta^{-1}\,(c_5c_4)^6\,(c_5c_4)^{-2}\right)^3 \quad \text{(chain relation } C_2\text{)} \\
\equiv \;& \left((c_1c_2)^2\,\delta^{-1}\,(c_5c_4)^4\right)^3 \quad \text{(cancelation)} \\
\equiv \;& \left((c_1c_2)^2\,xc_3c_1^{-2}c_5^{-2}\,(c_5c_4)^4\right)^3 \quad \text{(lantern relation)} \\
\equiv \;& \cdots\cdots \qquad\qquad (4.2) \\
\equiv \;& (c_1c_2xc_3c_4c_5c_5c_4c_5c_4)^3 \quad \text{(commutativity, braid relations)}
\end{aligned}$$

Cancelations do not change the signature and commutativity and braid relations have zero signature ([1], Proposition 3.6); therefore we have

$$\begin{aligned}\sigma(X_2) &= I(C_2) - I(C_2) + 3I(C_2) + 3I(L) \\ &= -7 - (-7) + 3(-7) + 3(+1) \\ &= -18\end{aligned}$$

Next, we compute $\sigma(X_3)$. Let C_2 denote either of the two chains $(c_1c_2)^6\,\delta_1^{-1}$ or $(c_7c_6)^6\,\delta_2^{-1}$ of length 2 and C_3 denote the chain $(e_1c_4f_1)^4\,\delta_1^{-1}\delta_2^{-1}$ of length 3 in $\mathcal{M}_3$. Then construction of the word in 3.2 gives

$$\begin{aligned}
& C_2 \cdot C_3^{-1} \cdot C_2 \\
= \; & (c_1c_2)^6\,\delta_1^{-1}\delta_1\,(e_1c_4f_1)^{-4}\,\delta_2\,\delta_2^{-1}\,(c_7c_6)^6 \\
\equiv \; & \left((c_1c_2)^2\,(e_1c_4f_1c_4)^{-1}\,(c_7c_6)^2\right)^3 \quad \text{(commutativity and cancelations)} \\
\equiv \; & \left((c_1c_2)^2\,(e_1c_4f_1c_4)^{-1}\,(e_1c_4f_1c_4)^3\,\delta_1^{-1}\delta_2^{-1}\,(c_7c_6)^2\right)^3 \quad \text{(chain relation } C_3\text{)} \\
\equiv \; & \left((c_1c_2)^2\,\delta_1^{-1}\,(e_1c_4f_1c_4)^2\,\delta_2^{-1}\,(c_7c_6)^2\right)^3 \quad \text{(commutativity and cancelations)} \\
\equiv \; & \left((c_1c_2)^2\,x_1c_3c_1^{-2}e_1^{-1}f_1^{-1}\,(e_1c_4f_1c_4)^2\,x_2c_5e_1^{-1}f_1^{-1}c_7^{-2}\,(c_7c_6)^2\right)^3 \quad \text{(2 lantern relations } L\text{)} \\
\equiv \; & \cdots\cdots \\
\equiv \; & (c_1c_2x_1c_3rc_8c_8c_4x_2c_5c_6c_7)^3 = 1 \quad \text{(commutativity, braid relations)}
\end{aligned}$$

Keeping track of the relations that are used in the process we obtain

$$\begin{aligned}\sigma(X_3) &= I(C_2) - I(C_3) + I(C_2) + 3I(C_3) + 3I(L) + 3I(L) \\ &= -7 - (-6) + (-7) + 3(-6) + 3(+1) + 3(+1) \\ &= -20\end{aligned}$$

Here we also used the fact that $(e_1c_4f_1)^4 = (e_1c_4f_1c_4)^3$.

We compute $\sigma(X_4)$ last. Following its construction in 3.3 we obtain

$$\begin{aligned}
& C_2 \cdot C_2 \cdot C_2 \cdot E^{-1} \\
= \; & (b_1a_1)^6\,\delta_1^{-1}\,(b_2a_2)^6\,\delta_2^{-1}\,(b_3a_3)^6\,\delta_3^{-1}\,(\alpha_1\alpha_2\alpha_3d)^{-3}\,\delta_1\delta_2\delta_3 \\
\equiv \; & \left((b_1a_1)^2\,(b_2a_2)^2\,(b_3a_3)^2\,(\alpha_1\alpha_2\alpha_3d)^{-1}\right)^3 \quad \text{(commutativity and cancelations)} \\
\equiv \; & \left((b_1a_1)^2\,(b_2a_2)^2\,(b_3a_3)^2\,\delta_1^{-1}\delta_2^{-1}\delta_3^{-1}\,(\alpha_1\alpha_2\alpha_3d)^3\,(\alpha_1\alpha_2\alpha_3d)^{-1}\right)^3 \quad \text{(star relation } E\text{)} \\
\equiv \; & \left((b_1a_1)^2\,(b_2a_2)^2\,(b_3a_3)^2\,\delta_1^{-1}\delta_2^{-1}\delta_3^{-1}\,(\alpha_1\alpha_2\alpha_3d)^2\right)^3 \quad \text{(commutativity and cancelations)} \\
\equiv \; & \Big((b_1a_1)^2\,(b_2a_2)^2\,(b_3a_3)^2\,x_1c_1a_1^{-1}a_1^{-1}\alpha_1^{-1}\alpha_2^{-1}x_2c_2a_2^{-1}a_2^{-1}\alpha_2^{-1}\alpha_3^{-1} \\
& x_3c_3a_3^{-1}a_3^{-1}\alpha_3^{-1}\alpha_1^{-1}\,(\alpha_1\alpha_2\alpha_3d)^2\Big)^3 \quad \text{(3 lantern relations } L\text{)} \\
\equiv \; & \cdots\cdots \\
\equiv \; & (a_1b_1a_2b_2a_3b_3x_1c_1x_2c_2x_3c_3rd)^3 = 1 \quad \text{(commutativity, braid relations)}
\end{aligned}$$

From this we obtain

$$\begin{aligned}\sigma(X_4) &= 3I(C_2) - I(E) + 3I(E) + 3I(L) + 3I(L) + 3I(L)\\ &= 3(-7) - (-5) + 3(-5) + 3(+1) + 3(+1) + 3(+1)\\ &= -22\end{aligned}$$

□

Consider now the fibrations $X_{g,k}$ given by the words (3.2),(3.5), and (3.17) which are obtained from (3.1), (3.4), and (3.16) by substituting k lantern relations.

Proposition 4.2. *The Euler characteristic and the signature of the manifold $X_{g,k}$ are given by $\sigma(X_{g,k}) = \sigma(X_g) + k$ and $\chi(X_{g,k}) = \chi(X_g) - k, g = 2, 3, 4$.*

Proof : The only substitutions used in (3.2),(3.5), and (3.17)that have nonzero signature are lantern relations. The rest of the modifications which result from commutativity and braid relations do not have nonzero contributions ([1], Proposition 3.6). Cancelations also do not effect the signature. Since the signature of each lantern relation is $+1$ half of the proof follows. The other half follows from (4.1) and the fact that each time we substitute a lantern relation the length of the word reduces by one. □

Remark 1. To be more specific about k we need to point out that $1 \leq k \leq 6$ for genus 2 and $1 \leq k \leq 3$ for genus $3, 4$. Therefore

$$-18 \leq \sigma(X_{2,k}) \leq -12, \quad -20 \leq \sigma(X_{3,k}) \leq -17, \quad -22 \leq \sigma(X_{4,k}) \leq -19$$

Remark 2. In order to see that we have a positive relator for each k we will show what the word for $X_{2,1}$ becomes, for example. $c_1c_2xc_3c_4c_3^{-2}khc_1c_4c_5c_4$ in the third line of (3.2) can be rewritten as $c_1c_2xc_4^{-1}c_3c_4c_3^{-1}c_4c_4^{-1}kc_4c_4^{-1}hc_4c_1c_5c_4$ and this becomes the positive relator $c_1c_2xmnp\,c_1c_5c_4$, where $m = c_4^{-1}c_3c_4c_3^{-1}c_4, n = c_4^{-1}kc_4$, and $p = c_4^{-1}hc_4$. Therefore the monodromy of $X_{2,1}$ is

$$c_1c_2xmnp\,c_1c_5c_4\left(c_1c_2xc_3c_4c_5c_5c_4c_5c_4\right)^2 = 1.$$

This is a fibration with $\sigma(X_{2,1}) = \sigma(X_2) + 1 = -17$ and $\chi(X_{2,1}) = \chi(X_2) - 1 = 25$.

Remark 3. An interesting thing to observe here is the effect of substituting a lantern relation into the monodromy of X_g on its homeomorphism invariants. Proposition 4.2 shows that it has the same effect on X_g as that of a rational blow-down operation on it. Therefore it's an interesting question to investigate whether or not X_g and $X_{g,k}\#k\overline{\mathbb{C}P}^2$ are diffeomorphic. See [2] for examples that answer this question in the negative.

Next in our list is the word (3.6) obtained from (3.5) by substituting m chain relations of length 3 into $X_{3,k}$, which will be denoted by $X_{3,k,m}, 1 \leq m \leq k \leq 3$. This notation does not reflect the length of the chain for the sake of simplicity. Note that chain substitution must follow a lantern substitution; therefore $m \leq k$.

Proposition 4.3. *$\sigma(X_{3,k,m}) = -20 + k - 6m$ and $\chi(X_{g,k,m}) = 28 - k + 10m$ for $1 \leq m \leq k \leq 3$.*

Proof : The signature of X_3 is -20 by Proposition (4.1) and the signature of $X_{3,k}$ was found to be $-20 + k$ in Proposition (4.2). Since $X_{3,k,m}$ is obtained from $X_{3,k}$

by substituting m chain relations of length 3 and C_3 has signature -6 (Proposition 3.10, [1]), we have

$$\sigma\left(X_{3,k,m}\right)=\sigma\left(X_{3,k}\right)+mI\left(C_3\right)=-20+k+m(-6), 1\leq m\leq k\leq 3.$$

Proposition (4.1) gives $\chi\left(X_3\right)=28$ and according to Proposition (4.2) $\chi\left(X_{g,k}\right)=28-k$. Since substitution of each C_3 results in increasing overall number of cycles by 10, its contribution to the Euler characteristic will be 10 according to (4.1). Therefore we have $\chi\left(X_{g,k,m}\right)=28-k+10m, 1\leq m\leq k\leq 3$.

□

Remark 4. Possible values for $\sigma\left(X_{3,k,m}\right)$ are $-23,-24,-25,-29,-30,-35$ and possible values for $\chi\left(X_{g,k,m}\right)$ are $35,36,37,45,46,55$.

Next, we will compute the signatures of the achiral Lefschetz fibrations (2.13) and (2.20), denote them by Z_g. Assume, first, g is even and greater than 7. Z_g has monodromy

$$c_1dx_1c_3r_1e_1e_1c_4x_2c_5f_2^{-1}W_6W_8\cdots W_gc_{2g-2}x_{g-1}c_{2g-1}c_{2g}c_{2g+1}c_{2g+1}c_{2g}c_{2g+1}c_{2g},$$

where $W_i=c_{2i-6}x_{i-3}c_{2i-5}r_{i-3}e_{i-3}e_{i-3}c_{2i-4}x_{i-2}c_{2i-3}f_{i-2}^{-1}, i=6,8,\ldots,g$.

From its construction in 2.1 we can see that this word originally contains two chains of length 2, one on each end, and $g-2$ chains of length 3 half of which are negatively oriented. Then we substituted $3(g-2)/2$ additional chains of length 3 in order to replace the negatively oriented ones by positive exponents. We also substituted 3 chains of length 2 for the same reason. These substitutions resulted in $3(g-1)$ separating negatively oriented boundary curves. Finally we introduced $3(g-1)$ lantern relations to eliminate them. The rest of the operations until we obtained (2.13) are cancelations, commutativity and braid relations, which have zero contribution to the signature. Combining all of that we can compute the signature of Z_g as

$$\sigma\left(Z_g\right)=I\left(C_2\right)-\frac{g-2}{2}I\left(C_3\right)+\frac{g-2}{2}I\left(C_3\right)-I\left(C_2\right)+\frac{3\left(g-2\right)}{2}I\left(C_3\right)+3I\left(C_2\right)+3\left(g-1\right)I\left(L\right)$$
$$=0+\frac{3\left(g-2\right)}{2}(-6)+3(-7)+3(g-1)(+1)=-6g-6$$

Suppose now that g is odd and greater than 6. This time Z_g is given by the monodromy

$$c_1dx_1c_3r_1c_{2g+2}c_{2g+2}c_4x_2c_5f_2^{-1}W_6W_8\cdots W_{g-1}W_g,$$

where $W_i=c_{2i-6}x_{i-3}c_{2i-5}r_{i-3}c_{2g+i-2}c_{2g+i-2}c_{2i-4}x_{i-2}c_{2i-3}f_{i-2}^{-1}, i=6,8,\ldots,g-1$ and
$W_g=c_{2g-6}x_{g-3}c_{2g-5}r_{g-3}c_{3g-2}c_{3g-2}c_{2g-4}x_{g-2}c_{2g-3}r_{g-2}f_{g-2}c_{3g-1}c_{2g-2}x_{g-1}c_{2g-1}c_{2g}c_{2g+1}$.
Using a similar argument we calculate the signature of Z_g as

$$\sigma\left(Z_g\right)=I\left(C_2\right)-\frac{g-1}{2}I\left(C_3\right)+\frac{g-3}{2}I\left(C_3\right)+I\left(C_2\right)+\frac{3\left(g-1\right)}{2}I\left(C_3\right)+3\left(g-1\right)I\left(L\right)$$

$$=-7-\frac{g-1}{2}\left(-6\right)+\frac{g-3}{2}\left(-6\right)+\left(-7\right)+\frac{3\left(g-1\right)}{2}\left(-6\right)+3\left(g-1\right)\left(+1\right)=-6g-2$$

Note that $\sigma\left(Z_g\right)=\sigma\left(X_g\right)$ for $g=2,3$. This is because the simplified form of the general construction leads to a positive relator. The existence of negative powers in the expression for higher genus, however, requires further substitution of lantern relations. We'll denote by Y_g a genus g Lefschetz fibration that is obtained from

either of the achiral Lefschetz fibrations (2.13) or (2.20) by substituting into them a number of lantern relations until a positive relator is obtained. In that regard the fibration given by (3.11) that is obtained from (3.7) via 3 lantern substitutions will be denoted by Y_4. If k additional lantern substitutions are made into these positive words then the resulting manifold will be denoted by $Y_{g,k}$. For example the positive relator (3.14) is denoted by $Y_{4,3}$ because it is obtained via 3 lantern substitutions into (3.11), which is equivalent to (3.9). We will now compute the signatures of $Y_g, Y_{g,k}$.

Proposition 4.4. $\sigma(Y_4) = -27, \sigma(Y_{4,k}) = -27 + k, \sigma(Y_5) = -29, \sigma(Y_6) = -30.$

Proof : Y_4 is obtained from Z_4 by substituting lantern relation 3 times; therefore

$$\sigma(Y_4) = \sigma(Z_4) + 3(+1) = -6 \cdot 4 - 6 + 3 = -27.$$

Similarly $Y_{4,k}$ is obtained from Y_4 by substituting k lantern relations,$1 \leq k \leq 3$; therefore

$$\sigma(Y_{4,k}) = \sigma(Y_4) + k(+1) = -27 + k.$$

Y_5 is obtained from Z_5 by substituting lantern relation 3 times; therefore

$$\sigma(Y_5) = \sigma(Z_5) + 3(+1) = -6 \cdot 5 - 2 + 3 = -29.$$

A careful analysis shows that Y_6 is obtained from Z_6 by substituting lantern relation 12 times (3 for each of the negative powers $f_2^{-1}, f_4^{-1}, f_3^{-1}, f_1^{-1}$); therefore

$$\sigma(Y_6) = \sigma(Z_6) + 12(+1) = -6 \cdot 6 - 6 + 12 = -30.$$

$\square$

We summarize what we found in the following table, which includes the Euler characteristic χ, signature σ, the holomorphic Euler characteristic $\chi_h = \frac{1}{4}(\sigma + \chi)$, and the self-intersection of the first Chern class $c_1^2 = 3\sigma + 2\chi$. The latter two are defined for manifolds having almost complex structure and symplectic Lefschetz fibrations are known to possess that.

	χ	σ	χ_h	c_1^2	π_1	
X_2	26	-18	2	-2	1	
$X_{2,k}$	$26-k$	$-18+k$	2	$-2+k$	1	$k=1,\dots,5$
$X_{2,6}$	20	-12	2	4	$\mathbb{Z}_3$	
X_3	28	-20	2	-4	1	
$X_{3,k}$	$28-k$	$-20+k$	2	$-4+k$	1	$k=1,2,3$
$X_{3,k,m}$	$28-k+10m$	$-20+k-6m$	$2+m$	$-4+k+2m$	1	$m,k=1,2,3,m\leq k$
X_4	30	-22	2	-6	1	
$X_{4,k}$	$30-k$	$-22+k$	2	$-6+k$	1	$k=1,2,3$
Y_4	39	-27	3	-3	1	
$Y_{4,k}$	$39-k$	$-27+k$	3	$-3+k$	1	$k=1,2,3$
Y_5	41	-29	3	-5	1	
Y_6	46	-30	4	2		

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Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan

E-mail address: `endo@math.sci.osaka-u.ac.jp`

Department of Mathematics and Computer Science, St. Louis University, MO, USA

E-mail address: `ygurtas@slu.edu`